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Existence and construction of neutrino fields with geodesic and shear-free rays and of other zero rest-mass fields in curved space–time

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Abstract. The Newman–Penrose formalism allows a unified treatment of unquantized Weyl neutrino fields with geodesic and shear-free rays ((κ, σ) fields) and of other zero rest-mass fields of type null with different spin. Taking Robinson's theorem about Maxwell null fields as a starting point, the necessary and sufficient conditions for a space–time to admit a neutrino (κ, σ) field or another null-type zero rest-mass field are given. In addition, a method of construction is developed which relates null-type fields and neutrino (κ, σ) fields with other null-type fields of different spin or, in an empty space–time, with the background conformal tensor.

1. Introduction

Any quantum field theory is based on an extensive knowledge of the classical fields which are to be quantized. Calculations make use of special solutions of the classical field equations. This justifies a detailed study of quantum mechanical field equations in the 'classical field approach' (i.e. on the level of unquantized fields) in flat and—with regard to the gravi–quantum mechanical interaction—in curved space–time as well.

Apart from the Weyl field (spin $s=\frac{1}{2}$, neutrinos) there are two other physically relevant fields describing particles of zero rest mass: the source-free Maxwell field ($s = 1$, photons) and the conformal tensor field of empty space–times ($s = 2$, gravitons). The two-spinor calculus allows a unified treatment of these and other zero rest-mass fields with spin $s \neq 0$ in flat and curved space–time. The field is thereby represented by a totally symmetric spinor $\Gamma^{A\ldots M}$ with $2s$ indices of one sort. The source-free field equations corresponding to the Weyl–Dirac–Pauli–Fierz equations in curved space-time† are

$$\Gamma^{A\ldots M}{}_{\|M\dot{X}}=0. \tag{1}$$

Our notation follows Pirani (1965).

The totally symmetric spinor $\Gamma^{A\ldots M}$ can be reduced to the form $\Gamma^{A\ldots M}=\kappa^{(A}\lambda^{B}\ldots\mu^{M)}$, where the spinors of rank one define $2s$ light-like directions called principal null directions. The following special case will be discussed below.

† For treatment in flat space–time see Roman (1960) and Umezewa (1956). It can easily be generalized to curved space–time by using covariant spinor derivatives instead of partial derivatives (Penrose 1965). For the connexion with the tensorial representation $F^{\alpha\beta}$ of the Maxwell field and $C^{\alpha\beta\gamma\delta}$ of the conformal tensor field see Pirani (1965).

Definition. A zero rest-mass field with spin $s \geq 1$ is said to be of *type null*, if its principal null directions coincide, i.e. if $\Gamma^{A...M}$ is of the form

$$\Gamma^{A...M} = \lambda^A \ldots \lambda^M \tag{2}$$

with some λ^A. The congruence of null vectors $l^\alpha \leftrightarrow \lambda^A \lambda^A$ is called *principal null congruence* (PNC).

We note that for the conformal tensor and the Maxwell field the null-type field means radiation field, while the spin-$\frac{1}{2}$ neutrino field Γ^M is automatically of type null. The corresponding field equation is the *Weyl equation*

$$\Gamma^M{}_{\|M\dot{X}} = 0. \tag{3}$$

We introduce a field of spin frames $\{o^A, \iota^A\}$ with $o_A \iota^A = 1$ and choose o^A to be everywhere parallel to λ^A:

$$\Gamma^{A...M} = \Phi o^A \ldots o^M. \tag{4}$$

This *adapted spin frame* is not uniquely defined. There is still the freedom of the following spin transformation (null rotation) which preserves the alignment of o^A:

$$\begin{aligned} o^A &\to o^{A*} = R^{1/2} e^{\frac{1}{2}iS} o^A \\ \iota^A &\to \iota^{A*} = \frac{1}{R^{1/2}} e^{-\frac{1}{2}iS} (\iota^A + R\bar{T} o^A) \end{aligned} \tag{5}$$

with R, S real, $R > 0$, T complex.

With respect to an adapted but otherwise arbitrary spin frame, the Weyl equation (3) takes the Newman–Penrose (NP) form†

$$D\Phi = (\rho - \epsilon)\Phi, \qquad \delta\Phi = (\tau - \beta)\Phi \tag{6}$$

while for null fields with $s \geq 1$ the field equations (1) become

$$\kappa\Phi = 0, \qquad \sigma\Phi = 0, \tag{7}$$

$$D\Phi = (\rho - 2s\epsilon)\Phi, \qquad \delta\Phi = (\tau - 2s\beta)\Phi. \tag{8}$$

The PNC of a null-type zero rest-mass field of spin $s \geq 1$ is therefore geodesic ($\kappa = 0$) and shear free ($\sigma = 0$). For the generic neutrino field neither κ nor σ vanishes (cf. Kuchowicz 1974, for a review). We call a neutrino field with geodesic and shear-free four-flux congruence a (κ, σ) field.

Neutrino (κ, σ) fields (spin $s = \frac{1}{2}$) and null-type zero rest-mass fields of spin $s \geq 1$ have the dynamical equations (8) in common. This makes a unified treatment possible.

The aim of this paper is twofold:

(i) Study of the necessary and sufficient conditions for a space–time to admit a neutrino (κ, σ) field or another null-type zero rest-mass field. For null-type Maxwell fields, Robinson's theorem (Robinson 1961) already provides the answer. Taking this as starting point, the unified treatment enables an appropriate generalization to the remaining fields of different spin (§ 2).

(ii) Description of a method to construct test fields out of other test fields with different spin or out of the conformal tensor field of the background space–time (§ 3).

† We assume the reader to be familiar with the NP formalism (Newman and Penrose 1962).

The results are equally valid for fields which contribute to the curvature of space–time through their energy–momentum tensors, as well as for fields which do not.

2. Existence and properties of solutions

As a first step, to study the necessary conditions, we suppose that a space–time admits a solution of (8). The vanishing of κ and σ of the PNC then implies, according to the NP Ricci identity (NP (4.2*b*))†,

$$D\sigma - \delta\kappa = \sigma(\rho+\bar{\rho}) + \sigma(3\epsilon - \bar{\epsilon}) - \kappa(\tau - \bar{\pi} + \bar{\alpha} + 3\beta) + \Psi_0 \tag{9}$$

that with respect to the adapted spin frame the component Ψ_0 of the conformal tensor of the underlying space–time vanishes:

$$\Psi_0 = 0. \tag{10}$$

Information about the component Ψ_1 can be obtained in combining the field equations (8), the commutation relation (NP (4.4)) (using $\kappa = 0$, $\sigma = 0$)

$$(\delta D - D\delta)\Phi = [(\bar{\alpha} + \beta - \bar{\pi})D - (\bar{\rho} + \epsilon - \bar{\epsilon})\delta]\Phi \tag{11}$$

and the Ricci identities (NP (4.2*c*, 4.2*e*, 4.2*k*))

$$D\tau = \rho(\tau + \bar{\pi}) + \tau(\epsilon - \bar{\epsilon}) + \Psi_1 + \Phi_{01} \tag{12}$$

$$D\beta - \delta\epsilon = \beta(\bar{\rho} - \bar{\epsilon}) - \epsilon(\bar{\alpha} - \bar{\pi}) + \Psi_1 \tag{13}$$

$$\delta\rho = \rho(\bar{\alpha} + \beta) + \tau(\rho - \bar{\rho}) - \Psi_1 + \Phi_{01}. \tag{14}$$

Substituting (8) into (11) we find with (12)–(14)

$$(s-1)\Psi_1 = 0. \tag{15}$$

We therefore have the following necessary conditions for the integration of the field equations:

If a space–time admits a neutrino (κ, σ) field ($s = \frac{1}{2}$) or a null-type zero rest-mass field (spin $s \geqslant 1$), then

(*a*) the PNC of this field coincides with a PNC of the conformal tensor of the space–time ($\Psi_0 = 0$), which accordingly must also be geodesic and shear free;

(*b*) Furthermore, for spin $s \neq 1$ (i.e. apart from the Maxwell field), this null congruence is a repeated PNC of the conformal tensor ($\Psi_1 = 0$) and the space–time must therefore be algebraically special‡.

For Maxwell fields ($s = 1$) the necessary conditions for the existence of null-type solutions are well known. Robinson's theorem (Robinson 1961) states: 'A congruence of null vectors k^α is geodesic and shear free if and only if there exists a non-trivial null-type solution of the source-free Maxwell equations with a PNC parallel to k^α'. Taking this theorem as a starting point, the similarity of the field equations enables us to prove the sufficiency of the conditions above for the remaining rest-mass fields in question.

† We refer to equation (4.2*b*) of Newman and Penrose (1962) by NP (4.2*b*) and so on.

‡ We mention that for a geodesic neutrino field ($\kappa = 0$) with shear ($\sigma \neq 0$) neither the component Ψ_0 nor the component Ψ_1 of the conformed tensor vanishes necessarily.

To do so, we eliminate in (8) the spin coefficients ϵ and β, and hence the spin value s, in changing to another adapted spin frame by means of a spin transformation (5) with $T=0$. This induces the following transformation of ϵ and β:

$$\begin{aligned}\epsilon \to \epsilon^* &= R(\epsilon + \tfrac{1}{2}DU)\\ \beta \to \beta^* &= e^{iS}(\beta + \tfrac{1}{2}\delta U)\end{aligned} \tag{16}$$

with $U = \ln R + iS$. We therefore must choose R and S to satisfy

$$DU = -2\epsilon, \qquad \delta U = -2\beta. \tag{17a, b}$$

The integrability conditions of these differential equations are obtained by applying the commutation relation (11). The Ricci identity (13) then shows that these conditions are satisfied provided that κ, σ and Ψ_1 vanish. For such a congruence and space–time the differential equations (17) can be integrated†. A null rotation with the resulting R and S leads to vanishing ϵ^* and β^*, and the field equations (8) take, for all fields independent of the respective spin s, the same form‡

$$D^*\Phi^* = \rho^*\Phi^*, \qquad \delta^*\Phi^* = \tau^*\Phi^* \tag{18}$$

where Φ^* and Φ are because of $\Phi = \Gamma_{A\ldots M}\iota^A \ldots \iota^M \to \Phi^* = \Gamma_{A\ldots M}\iota^{A^*} \ldots \iota^{M^*}$ related by

$$\Phi = e^{-isS}R^{-s}\Phi. \tag{19}$$

Thus with regard to this special spin frame, solution of the field equations with different spins and coinciding PNC are represented by the same solutions Φ^*.

Given in an algebraically special space–time a null congruence with $\kappa = 0$, $\sigma = 0$ and oriented so that $\Psi_1 = 0$, we know from Robinson's theorem that there exists an aligned null-type Maxwell solution. Furthermore, by a null rotation, the field equations can be given the form (18). Therefore there exists at least one solution Φ^* of (18). As shown above this is also a solution of the field equations for the other spin values $s \neq 1$. Accordingly we have, as a converse of the theorem above, the following result regarding the existence of solutions of the field equations (8):

> if in an algebraically special space–time ($\Psi_0 = \Psi_1 = 0$) the repeated PNC of the conformal tensor is geodesic and shear free ($\kappa = 0$, $\sigma = 0$), then there exists with the same PNC a neutrino (κ, σ) solution ($s = \frac{1}{2}$) and, for all spin values $s \geqslant 1$, a null-type zero rest-mass solution.

3. Construction of solutions

The unification (18) enables us to construct from a solution Φ of (8) for a particular spin value $s_1 \neq 1$ (denoted by $\Phi[s_1]$) other solutions of different spin s_2 (denoted by $\Phi[s_2]$). To do so for the spin value $s_1 = 1$ as well, i.e. if the original field is a null-type Maxwell field, we have to assume additionally that its PNC coincides with a repeated PNC of the conformal tensor of the respective space–time. Under this assumption we have in an

† In order to obtain the solution $U = \ln R + iS$ in practice, we adapt coordinates to the null tetrad field with $k^\alpha \leftrightarrow o^A o^{\dot{A}}$ and solve (17b) in the hypersurface of a fixed value of the k^α coordinate. The corresponding null rotation leads to $\beta^* = 0$ in these points. Solution of (17a) along the k^α lines enables $\epsilon^* = 0$ everywhere and completes $U(x)$. The vanishing of κ, σ and Ψ_1 then finally implies $\kappa^* = \sigma^* = \Psi_1^* = 0$, so that the propagation equation (13) guarantees $\beta^* = 0$ everywhere.

‡ Note the corresponding transformation behaviour $D \to D^* = RD$, $\rho \to \rho^* = R\rho$, $\delta \to \delta^* = e^{iS}\delta$, $\tau \to \tau^* = e^{iS}\tau$.

adapted spin frame $\kappa = \sigma = \Psi_0 = \Psi_1 = 0$ and can perform a null rotation with R and S of (17) leading to

$$\Phi^*[s_1] = e^{-is_1S} R^{-s_1} \Phi[s_1]. \tag{20}$$

As shown above, this Φ^* is also a solution of a field with spin s_2

$$\Phi^*[s_1] = \Phi^*[s_2] = e^{-is_2S} R^{-s_2} \Phi[s_2]. \tag{21}$$

With regard to the original spin frame, solutions of different spin s_1 and s_2 and same PNC are therefore related by

$$\Phi[s_2] = e^{i(s_2-s_1)S} R^{(s_2-s_1)} \Phi[s_1] \tag{22}$$

with R and S being the solutions of (17).

Apart from this method of construction which relates null-type solutions with null-type solutions of different spin or neutrino (κ, σ) fields respectively, there is still another one which establishes a connection with the conformal tensor of the underlying space–time. We restrict to vacuum space–times and assume additionally that the space–time is algebraically special

$$\Psi_0 = 0, \qquad \Psi_1 = 0 \tag{23}$$

and that the repeated PNC is geodesic and shear free

$$\kappa = 0, \qquad \sigma = 0. \tag{24}$$

In the following we treat space–times of different Petrov type separately.

3.1. Petrov type D or II

In this case we have $\Psi_2 \neq 0$ and the vacuum Bianchi identities (NP (4.5)) contain, using (23) and (24),

$$D\Psi_2 = 3\rho\Psi_2, \qquad \delta\Psi_2 = 3\tau\Psi_2. \tag{25}$$

The assumptions (23) and (24) also ensure that the integrability conditions of (17) are fulfilled, so that we can again perform the null rotation with R and S of (17) leading to Bianchi identities (cf. the last footnote):

$$D^*\Psi_2^* = 3\rho^*\Psi_2^*, \qquad \delta^*\Psi_2^* = 3\tau^*\Psi_2^* \tag{26}$$

with $\Psi_2^* = \Psi_2$ and to field equations of the form (18). Comparison shows that we can construct a special solution Φ^* by

$$\Phi^* = (\Psi_2^*)^{1/3}. \tag{27}$$

Transforming back to the original spin frame with (20) we have that

$$\Phi[s] = e^{isS} R^s \Psi_2^{1/3} \tag{28}$$

is a special (κ, σ) neutrino field ($s = \frac{1}{2}$) or null-type zero rest-mass field of spin $s = 1$ propagating along the PNC.

3.2. Petrov type III

Now we have $\Psi_2 = 0$ and $\Psi_3 \neq 0$ and the vacuum Bianchi identities (NP (4.5)) contain, using (23) and (24),

$$D\Psi_3 = 2(\rho - \epsilon)\Psi_3, \qquad \delta\Psi_3 = 2(\tau - \beta)\Psi_3. \tag{29}$$

Comparision with (6) shows that Ψ_3 leads to the special (κ, σ) neutrino solution

$$\Phi[s = \tfrac{1}{2}] = \Psi_3^{1/2} \tag{30}$$

out of which we can construct null-type solutions with arbitrary spin $s \geqslant 1$ using (22):

$$\Phi[s] = e^{i(s-\frac{1}{2})S} R^{(s-\frac{1}{2})} \Psi_3^{1/2}. \tag{31}$$

3.3. Petrov type N

In this case the conformal tensor itself is a source-free null-type zero rest-mass field with spin $s = 2$ and $\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0$, $\Psi_4 \neq 0$. A special null-type solution of other spin $s \geqslant 1$ or a special neutrino (κ, σ) field $(s = \frac{1}{2})$ is according to (22) given by

$$\Phi[s] = e^{i(s-2)S} R^{(s-2)} \Psi_4. \tag{32}$$

It is the advantage of the two methods stated above, that they enable us to use the vast literature concerning null-type Maxwell fields or vacuum conformal tensor fields in various space–times, in order to find other zero rest-mass fields explicitly or to study their properties in general.

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